Rendering Contact Binary Stars

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In this article we develop a mathematical model for creating computer generated renderings of two stars in a binary system that are in contact with one another. A binary star system consists of two stars that are gravitationally bound. Each star in a binary system can be pictured as being surrounded by a tear-shaped zone of gravitational influence, the Roche lobe\(^1\). Any material within the Roche lobe of a star can be considered to be part of that star. During evolution, one member of the binary star can expand so that it overflows its own Roche lobe, and begins to transfer matter onto the other star.

A detached binary is a kind of binary star where each component is completely within its Roche lobe. The stars have no major effect on each other, and essentially evolve separately. Most binaries belong to this class.

A semidetached binary is a kind of binary star in which one of the components fills its Roche lobe and the other does not. The star that fills its Roche lobe can transfer material to the other star creating an accretion disk.

A contact binary has both components of the system filling their Roche lobes. The uppermost part of the stellar atmospheres forms a common envelope that surrounds both stars.

In a previous article\(^1\), we were able show how to find the stellar radius, \(r\), as a function of the angle from the pole, \(\theta\), and azimuthal angle, \(\phi\), for each star. This method only works for detached and semidetached systems. For contact binaries the radius is, of course, mathematically undefined near the point between the two stars. However, if we instead use cylindrical geometry we can avoid this problem.

Consider two stars separated by a distance \(R\) that rotate around one another with an angular velocity of \(\omega\) about an axis that is parallel to the \(z\)-axis.

\[ \text{Figure 2: Geometry of a Binary Star} \]

\[ \text{Figure 1: Types of Binaries} \]
The shape of the contact binary can be determined by looking at the forces on a small element of mass, \( m \), within the system.

\[
\sum \vec{F} = m\vec{a}
\]

\[
\vec{F}_{\text{pressure}} + \vec{F}_{\text{gravity}} = -m\omega^2\rho'
\]

where \( \rho' \) is the axial distance from the axis of rotation. The prime (') is used here for coordinates relative to the axis of rotation and terms that do not have primes will be used for coordinates relative to the center of \( m_1 \). The first term above is the pressure force (or pressure-gradient force) and is defined to be the force due to differences of pressure within a fluid mass. The pressure force per unit volume is equal to the negative gradient of the pressure. This gives us

\[
\vec{F}_{\text{pressure}} = -\nabla P = -\frac{m}{\rho_m} \nabla P
\]

where \( \rho_m \) is the mass density of the volume \( V \) of mass \( m \). Note that the rho here is density and should not be confused with the rho vector in Equation 1. Notice that the vector sum of two of the terms in Equation (1) can be expressed as the negative gradient of a potential \( \Phi \).

\[
\vec{F}_{\text{gravity}} + m\omega^2\rho' = -\frac{Gm_1m}{r_1^2} \hat{r}_1 - \frac{Gm_2m}{r_2^2} \hat{r}_2 + m\omega^2\rho' = -m\nabla \Phi
\]

where \( r_1 \) and \( r_2 \) are the distances of small element of mass, \( m \), from the centers of \( m_1 \) and \( m_2 \) respectively. From Equations (1), (2) and (3) we see that

\[
\nabla P = -\rho_m \nabla \Phi
\]

which implies that \( \nabla P \) and \( \nabla \Phi \) are parallel. If we take the curl of Equation (4) we find

\[
\nabla \times \nabla P = \nabla \times (-\rho_m \nabla \Phi) = -\rho_m \nabla \times \nabla \Phi - \nabla (\rho_m \times \nabla \Phi).
\]

The curl of gradient of any function is zero. Therefore,

\[
\nabla \rho_m \times \nabla \Phi = 0
\]

from which it follows that \( \nabla \rho_m \) is parallel to both \( \nabla P \) and \( \nabla \Phi \). This circumstance implies that surfaces of constant density, pressure and \( \Phi \) coincide. So if we find shapes corresponding to constant \( \Phi \), then we have found the possible shapes of the stars of a binary system. From equation (3) we find that

\[
\Phi = -\frac{Gm_1}{r_1} - \frac{Gm_2}{r_2} - \frac{1}{2} \omega^2 \rho'^2.
\]
The coordinates of the small element of mass, \( m \), can be expressed in Cartesian coordinates \((x, y, z)\) or cylindrical coordinates \((\rho, \phi, z)\).

\[
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z \\
\rho &= \sqrt{x^2 + y^2}
\end{align*}
\]  

(6)

Notice in Figure 2 that the center of mass lies on the \( z \)-axis and its distance from \( m_1 \) is

\[
Z = \sum \frac{m_i z_i}{\sum m_i} = \frac{m_2 R}{m_1 + m_2} = \frac{qR}{1 + q}
\]  

(7)

where \( q = m_2/m_1 \).

Now we can find \( r_1, r_2 \) and \( \rho' \) in terms of the cylindrical coordinates relative to our origin in the center of \( m_1 \).

\[
\begin{align*}
r_1 &= \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} \\
r_2 &= \sqrt{x^2 + y^2 + (z - R)^2} = \sqrt{\rho^2 + (z - R)^2} \\
\rho' &= \sqrt{y^2 + (z - \bar{z})^2} = \sqrt{\rho^2 \sin^2 \phi + (z - \bar{z})^2}
\end{align*}
\]  

(8)

Using Equations (5) and (8), we get

\[
\Psi(\rho, \phi, z) = -\frac{Gm_1}{\sqrt{\rho^2 + z^2}} - \frac{Gm_2}{\sqrt{\rho^2 + (z - R)^2}} - \frac{1}{2} \frac{G(m_1 + m_2)}{R^3} \left( \rho^2 \sin^2 \phi + \left( z - \frac{qR}{1 + q} \right)^2 \right).
\]  

(9)

From Kepler’s third law we have

\[
\omega^2 = \frac{G(m_1 + m_2)}{R^3}.
\]  

(10)

Using Equations (9), (7), and (10), we get

\[
\Psi(\rho, \phi, z) = -\frac{Gm_1}{\sqrt{\rho^2 + z^2}} - \frac{Gm_2}{\sqrt{\rho^2 + (z - R)^2}} - \frac{1}{2} \frac{G(m_1 + m_2)}{R^3} \left( \rho^2 \sin^2 \phi + \left( z - \frac{qR}{1 + q} \right)^2 \right)
\]  

(11)

We can now define a new potential function as Kopal\(^3\) did in 1959.

\[
\Omega = -\frac{R\Psi}{Gm_1} = \frac{1}{2} \frac{q^2}{1 + q}
\]  

(12)
Using Equations (11) and (12) we get
\[
\Omega(\rho, \phi, z) = \frac{R}{\sqrt{\rho^2 + z^2}} + \frac{qR}{\sqrt{\rho^2 + (z-R)^2}} + \frac{1}{2} \left( \frac{1+q}{R^2} \right) \left( \rho^2 \sin^2 \phi + z^2 \frac{2zqR}{1+q} \right) \tag{13}
\]

We can express \( \rho \) and \( z \) as fractions of \( R \) as the unitless terms
\[
\tilde{\rho} = \frac{\rho}{R} \quad \text{and} \quad \tilde{z} = \frac{z}{R}. \tag{14}
\]

Using Equations (13) and (14) we get
\[
\Omega(\tilde{\rho}, \phi, \tilde{z}) = \frac{1}{\sqrt{\tilde{\rho}^2 + \tilde{z}^2}} + \frac{q}{\sqrt{\tilde{\rho}^2 + (\tilde{z} - 1)^2}} + \frac{1}{2} \left( \frac{1+q}{\tilde{z}^2} \right) \left( \tilde{\rho}^2 \sin^2 \phi + \tilde{z}^2 \frac{2\tilde{z}q}{1+q} \right) \tag{15}
\]

At the pole of \( m_1 \), \( \phi=0 \), \( \tilde{z}=0 \), \( \tilde{\rho} = \tilde{r}_{\text{pole}} \) and
\[
\Omega(\tilde{r}_{\text{pole}}, 0, 0) = \frac{1}{\tilde{r}_{\text{pole}}} + \frac{q}{\sqrt{\tilde{r}_{\text{pole}}^2 + 1}} = \text{constant}. \tag{16}
\]

So to find the surface of the star we only need to find \( \tilde{\rho} \) for each \( \phi \) and \( \tilde{z} \) using
\[
\frac{1}{\tilde{r}_{\text{pole}}} + \frac{q}{\sqrt{\tilde{r}_{\text{pole}}^2 + 1}} = \frac{1}{\sqrt{\tilde{\rho}^2 + \tilde{z}^2}} + \frac{q}{\sqrt{\tilde{\rho}^2 + (\tilde{z} - 1)^2}} + \frac{1}{2} \left( \frac{1+q}{\tilde{z}^2} \right) \left( \tilde{\rho}^2 \sin^2 \phi + \tilde{z}^2 \frac{2\tilde{z}q}{1+q} \right) \tag{17}
\]
or if we solve for first \( \tilde{\rho} \) above we get
\[
\tilde{\rho} = \sqrt{\frac{1}{\tilde{r}_{\text{pole}}} + \frac{q}{\sqrt{\tilde{r}_{\text{pole}}^2 + 1}} - \frac{q}{\sqrt{\tilde{\rho}^2 + (\tilde{z} - 1)^2}} - \frac{1}{2} \left( \frac{1+q}{\tilde{z}^2} \right) \left( \tilde{\rho}^2 \sin^2 \phi + \tilde{z}^2 \frac{2\tilde{z}q}{1+q} \right)^{-2} - \tilde{z}^2}. \tag{18}
\]

There is clearly not a simple analytical solution of \( \tilde{\rho} \) as a function of \( \phi \) and \( \tilde{z} \). However, there are ways that we can obtain the values of \( \tilde{\rho} \) to an arbitrary accuracy using iteration techniques. We can use an initial guess for \( \tilde{\rho} \) on the right hand side of Equation (18) and compute \( \tilde{\rho} \) on the left hand side. The new computed value of \( \tilde{\rho} \) can then be used on the right hand side as a better guess. Each iteration step will yield a more accurate value for \( \tilde{r} \). Only a few iterations are needed since the differences between the guess and the result decrease rapidly. A technique for using Equation 18 to create a computer generated rendering is outlined step-by-step on the following pages.
Step 1: We must first find where the system begins and ends along the z-axis. On the z-axis, $\bar{\rho} = 0$ and Equation 17 becomes

$$\frac{1}{\bar{r}_{\text{pole}}} + \frac{q}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} = \pm \frac{1}{\bar{z}} + \frac{q}{|\bar{z} - 1|} + \frac{1}{2}(1 + q)\left(\bar{z}^2 - 2\bar{z}q\right).$$  \hspace{1cm} (19)$$

Rearranging the terms we get

$$\frac{1}{\bar{z}} = -\left(\frac{1}{\bar{r}_{\text{pole}}} + \frac{q}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} - \frac{q}{|\bar{z} - 1|} - \frac{1}{2}(1 + q)\left(\bar{z}^2 - 2\bar{z}q\right)\right)$$

and

$$\frac{1}{\bar{z} - 1} = \frac{1}{q}\left(\frac{1}{\bar{r}_{\text{pole}}} + \frac{q}{\sqrt{\bar{r}_{\text{pole}}^2 + 1}} - \frac{1}{\bar{z}} - \frac{1}{2}(1 + q)\left(\bar{z}^2 - 2\bar{z}q\right)\right) .$$ \hspace{1cm} (21)

The roots of this equation can be found using the interactive technique described earlier. An initial guess of $(-\bar{\rho}_{\text{pole}})$ for $\bar{z}$ in Equation 20 will give the intersection point of $m_1$ on the z-axis. An initial value of $(\bar{R} + \frac{q}{\bar{\rho}_{\text{pole}}} \bar{z})$ for $\bar{z}$ in Equation 21 will give the intersection of $m_2$ on the z-axis after a few iterations. The distances that result from the iteration will be called $\bar{z}_{\text{min}}$ and $\bar{z}_{\text{max}}$ respectively. Note that $\bar{r}_{\text{back}} = \bar{z}_{\text{min}}$ and $\bar{r}_{\text{2back}} = (\bar{z}_{\text{max}} - \bar{R})$. For example, for $q=0.3$ and $\bar{r}_{\text{pole}} = 0.460$ we get $\bar{r}_{\text{back}} = 0.522$ and $\bar{r}_{\text{2back}} = 0.313$.

![Figure 3: Stellar Dimension Definitions](image)

Figure 3 defines a few terms that can help us describe the shape of a star. A fourth parameter, $r_{\text{side}}$, is perpendicular to $r_{\text{pole}}$, $r_{\text{back}}$, and $r_{\text{point}}$. These all can be found using spherical geometry. Note that $r_{\text{point}}$ is undefined for contact binary systems.
Excel worksheets, computer programs, and other examples can be found at the website below.

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http://midnightkite.com/binstar.htm

References

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